J. Differential Equations 249 (2010) 1179–1198

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Local strong solution to the compressible viscoelastic flow with large data

Xianpeng Hu *, Dehua Wang

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

ARTICLE INFO

Article history:
Received 20 January 2010
Revised 18 March 2010
Available online 9 April 2010

MSC:
35A05
76A10
76D03

Keywords:
Compressible viscoelastic flows
Strong solution
Existence
Uniqueness

ABSTRACT

The existence and uniqueness of local in time strong solution with large initial data for the three-dimensional compressible viscoelastic flow is established. The strong solution has weaker regularity than the classical solution. The Lax–Milgram theorem and the Schauder–Tychonoff fixed-point argument are applied.

1. Introduction

Elastic solids and viscous fluids are two extremes of material behavior. Viscoelastic flows show intermediate behavior with some remarkable phenomena due to their "elastic" nature. These fluids exhibit a combination of both fluid and solid characteristics, keep memory of their past deformations, and their behavior is a function of these old deformations. Viscoelastic flows have a wide range of applications and hence have received a great deal of interest. Examples and applications of viscoelastic flows include from oil, liquid polymers, mucus, liquid soap, toothpaste, clay, ceramics, gels, some types of suspensions, to bioactive fluids, coatings and drug delivery systems for controlled drug release, scaffolds for tissue engineering, and viscoelastic blood fluid flow past valves; see [6,11,32] for more applications. For the viscoelastic materials, the competition between the kinetic energy and the internal elastic energy through the special transport properties of their respective internal elastic vari-

* Corresponding author.
E-mail addresses: xih15@pitt.edu (X. Hu), dwang@math.pitt.edu (D. Wang).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.03.027

ables makes the materials more untractable in understanding their behavior, since any distortion of microstructures, patterns or configurations in the dynamical flow will involve the deformation tensor. For classical simple fluids, the internal energy can be determined solely by the determinant of the deformation tensor; however, the internal energy of complex fluids carries all the information of the deformation tensor. The interaction between the microscopic elastic properties and the macroscopic fluid motions leads to the rich and complicated rheological phenomena in viscoelastic flows, and also causes formidable analytic and numerical challenges in mathematical analysis. The equations of the compressible viscoelastic flows of Oldroyd type [25,26] in three spatial dimensions take the following form [9,20,27]:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1a}$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \nabla P(\rho) = \operatorname{div}\big(\rho \mathtt{F}\mathtt{F}^{\top}\big), \tag{1.1b}$$

$$\mathtt{F}_t + \mathbf{u} \cdot \nabla \mathtt{F} = \nabla \mathbf{u} \mathtt{F}, \tag{1.1c}$$

where ρ stands for the density, $\mathbf{u} \in \mathbb{R}^3$ the velocity, $\mathtt{F} \in M^{3\times 3}$ the deformation gradient, and $P(\rho)$ the pressure which is a strictly increasing convex function of the density. The notation $M^{3\times 3}$ means the set of all 3×3 matrices. The viscosity coefficients μ and λ satisfy the conditions that $3\mu + 2\lambda > 0$ and $\mu > 0$, which ensure that the operator $-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u}$ is a strongly elliptic operator. The symbol $\otimes$ denotes the Kronecker tensor product, $\mathtt{F}^{\top}$ means the transpose matrix of $\mathtt{F}$, and the notation $\mathbf{u} \cdot \nabla \mathtt{F}$ is understood to be $(\mathbf{u} \cdot \nabla)\mathtt{F}$. Usually, we refer Eq. (1.1a) as the continuity equation.

We are interested in the Cauchy problem of (1.1) with the initial condition:

$$(\rho, \mathbf{u}, \mathtt{F})(0, x) = (\rho_0, \mathbf{u}_0, \mathtt{F}_0)(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

The aim of this paper is to establish the local existence and uniqueness of strong solution to system (1.1) with large initial data in the three-dimensional space $\mathbb{R}^3$. By a *strong solution*, we mean a triplet $(\rho, \mathbf{u}, \mathtt{F})$ with $\mathbf{u}(t, \cdot) \in W^{2,q}$ and $(\rho(t, \cdot), \mathtt{F}(t, \cdot)) \in W^{1,q}$, $q > 3$ satisfying (1.1) almost everywhere with the initial condition (1.2). There have been many studies and rich results in the literature for the global existence of classical solutions (namely in H^3 or other functional spaces with much higher regularity) for the corresponding *incompressible* viscoelastic fluids, see [1,2,10,12–14,16,20,21] and the references therein. For the compressible viscoelastic flows (1.1), the global existence of classical solutions in H^3 with small perturbation near its equilibrium for (1.1) without the pressure term was studied in [15], and a local existence of strong solution near the equilibrium and a series of uniform estimates were obtained in [7]. One of the main difficulties in proving the global existence is the lacking of the dissipative estimates for the deformation gradient and the gradient of the density. To overcome this difficulty, for incompressible cases, authors in [14] introduced an auxiliary function to obtain the dissipative estimate for the classical solutions, while authors in [21] directly deal with the quantities such as $\Delta\mathbf{u} + \operatorname{div} \mathtt{F}$. Since we are concerned with the *strong solutions in* $W^{2,q}$ which have weaker regularity than the *classical solution in* H^3, we first linearize (1.1) and use the Lax–Milgram theorem to obtain the solution to the linearized system, then we apply the Schauder–Tychonoff fixed point theorem to obtain the strong solution of (1.1).

Using the standard notations $W^{s,q}(\mathbb{R}^3)$ ($H^s(\mathbb{R}^3)$ if $q = 2$) for the Sobolev spaces and setting $Q_T := [0, T] \times \mathbb{R}^3$ for any $T > 0$, we can state our result on the existence and uniqueness as follows.

Theorem 1.1. *Assume that*

$$\rho_0 \in W^{1,q}\big(\mathbb{R}^3\big) \cap H^1\big(\mathbb{R}^3\big), \qquad \mathbf{u}_0 \in H^2\big(\mathbb{R}^3\big), \qquad \mathtt{F}_0 \in W^{1,q}\big(\mathbb{R}^3\big) \cap H^1\big(\mathbb{R}^3\big),$$

for some $q \in (3, 6]$, *and for some positive constants* α, β, *and* r_0,

$$\alpha \leqslant \rho_0 \leqslant \beta, \quad \|\mathbf{u}_0\|_{H^2} + \|\rho_0\|_{W^{1,q}\cap H^1} + \|\mathtt{F}_0\|_{W^{1,q}\cap H^1} \leqslant r_0.$$

Then there are positive constants $\overline{T} = \overline{T}(r_0)$, $\alpha(\overline{T}, r_0, \alpha)$, and $\beta(\overline{T}, r_0, \beta)$, such that, the Cauchy problem (1.1)–(1.2) *has a unique strong solution* $(\rho, \mathbf{u}, \mathtt{F})$ *defined for* $(t, x) \in (0, \overline{T}) \times \mathbb{R}^3$*, satisfying*

$$\alpha(\overline{T}, r_0, \alpha) \leqslant \rho \leqslant \beta(\overline{T}, r_0, \beta);$$
$$\rho \in L^\infty\big(0, \overline{T}; W^{1,q}\big(\mathbb{R}^3\big) \cap H^1\big(\mathbb{R}^3\big)\big) \cap L^\infty(Q_{\overline{T}}); \qquad \partial_t \rho \in L^\infty\big(0, \overline{T}; L^q\big(\mathbb{R}^3\big)\big);$$
$$\mathbf{u} \in L^2\big(0, \overline{T}; W^{2,q}\big(\mathbb{R}^3\big) \cap H^2\big(\mathbb{R}^3\big)\big); \qquad \partial_t \mathbf{u} \in L^2\big(0, \overline{T}; H^1\big(\mathbb{R}^3\big)\big);$$
$$\mathtt{F} \in L^\infty\big(0, \overline{T}; W^{1,q}\big(\mathbb{R}^3\big) \cap H^1\big(\mathbb{R}^3\big)\big) \cap L^\infty(Q_{\overline{T}}); \qquad \partial_t \mathtt{F} \in L^\infty\big(0, \overline{T}; L^q\big(\mathbb{R}^3\big)\big).$$

The viscoelasticity system (1.1) can be regarded as a combination between the compressible Navier–Stokes equation with the source term $\operatorname{div}(\rho \mathtt{F}\mathtt{F}^\top)$ and Eq. (1.1c). As for the global existence of classical solutions of the small perturbation near an equilibrium for compressible Navier–Stokes equations, we refer the interested reader to [22,23] and the references cited therein. The global existence of strong solutions with small perturbations near an equilibrium for compressible Navier–Stokes equations was also discussed in [24,29]. Unlike the Navier–Stokes equations [4,17,18], the global existence of weak solutions to (1.1) with large initial data is still an outstanding open question. In this direction for the incompressible viscoelasticity, when the contribution of the strain rate (symmetric part of $\nabla \mathbf{u}$) in the constitutive equation is neglected, Lions and Masmoudi in [19] proved the global existence of weak solutions with large initial data for the Oldroyd model. Also Lin, Liu, and Zhang showed in [12] the existence of global weak solutions with large initial data for the incompressible viscoelasticity if the velocity satisfies the Lipschitz condition. When dealing with the global existence of weak solutions with large data in the compressible case, among all of difficulties, the rapid oscillation of the density and the non-compatibility between the quadratic form and the weak convergence are of the main issues. For the inviscid elastodynamics, see [8,30,31] and their references on the finite-time blow up and global existence of classical solutions.

This paper will be organized as follows. In Section 2, we will recall briefly the compressible viscoelastic system from some fundamental mechanical theory. In Section 3 and Section 4, we will give the proof of the main theorem (Theorem 1.1). More precisely, Section 3 is devoted to the local existence of the system (1.1) by the Lax–Milgram theorem and the Schauder–Tychonoff fixed-point argument, while the main goal of Section 4 is to prove the uniqueness of the solution obtained in Section 3.

2. Mechanical background of viscoelasticity

In this section, we discuss some background of viscoelasticity and some basic properties of system (1.1). First, we recall the definition of the deformation gradient $\mathtt{F}$. The dynamics of any mechanical problem with a velocity field $\mathbf{u}(x, t)$ can be described by the flow map, a time dependent family of orientation preserving diffeomorphisms $x(t, X)$, $0 \leqslant t \leqslant T$:

$$\begin{cases} \dfrac{d}{dt} x(t, X) = \mathbf{u}\big(t, x(t, X)\big), \\ x(0, X) = X. \end{cases} \tag{2.1}$$

The material point (labeling) X in the reference configuration is deformed to the spatial position $x(t, X)$ at time t, which is the observer's coordinate.

The deformation gradient $\widetilde{\mathtt{F}}$ is used to describe the changing of any configuration, amplification or pattern during the dynamical process, which is defined as

$$\widetilde{\mathtt{F}}(t, X) = \frac{\partial x}{\partial X}(t, X).$$

Notice that this quantity is defined in the Lagrangian material coordinate. Obviously it satisfies the following rule, by changing the order of the differentiation:

$$\frac{\partial \widetilde{\mathtt{F}}(t,X)}{\partial t} = \frac{\partial \mathbf{u}(t,x(t,X))}{\partial X}. \tag{2.2}$$

In the Eulerian coordinate, the corresponding deformation gradient $\mathtt{F}(t,x)$ will be defined as $F(t,x(t,X)) = \widetilde{\mathtt{F}}(t,X)$. Eq. (2.2), together with the chain rule and (2.1), yield the following equation:

$$\begin{aligned}
\partial_t \mathtt{F}\big(t,x(t,X)\big) + \mathbf{u}\cdot\nabla \mathtt{F}\big(t,x(t,X)\big) &= \partial_t \mathtt{F}\big(t,x(t,X)\big) + \frac{\partial \mathtt{F}(t,x(t,X))}{\partial x}\cdot\frac{\partial x(t,X)}{\partial t}\\
&= \frac{\partial \widetilde{\mathtt{F}}(t,X)}{\partial t} = \frac{\partial \mathbf{u}(t,x(t,X))}{\partial X}\\
&= \frac{\partial \mathbf{u}(t,x(t,X))}{\partial x}\frac{\partial x}{\partial X}\\
&= \frac{\partial \mathbf{u}(t,x(t,X))}{\partial x}\widetilde{\mathtt{F}}(t,X)\\
&= \nabla\mathbf{u}\cdot\mathtt{F},
\end{aligned}$$

which is exactly Eq. (1.1c). Here, and in what follows, we use the conventional notations:

$$(\nabla\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}, \qquad (\nabla\mathbf{u}\mathtt{F})_{i,j} = (\nabla\mathbf{u})_{ik}F_{kj}, \qquad (\mathbf{u}\cdot\nabla\mathtt{F})_{ij} = u_k\frac{\partial \mathtt{F}_{ij}}{\partial x_k},$$

and summation over repeated indices will always be well understood. In viscoelasticity, (1.1c) can also be interpreted as the consistency of the flow maps generated by the velocity field $\mathbf{u}$ and the deformation gradient $\mathtt{F}$.

The difference between fluids and solids lies in the fact that in fluids, such as Navier–Stokes equations [24], the internal energy can be determined solely by the determinant part of $\mathtt{F}$ (equivalently the density ρ, and hence, (1.1c) can be disregarded) and in elasticity, the energy depends on all information of $\mathtt{F}$.

In the continuum physics, if we assume that the material is homogeneous, the conservation laws of mass and of momentum become [3,14,31]

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{2.3}$$

and

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P(\rho) = \operatorname{div}\big((\det\mathtt{F})^{-1}S\mathtt{F}^\top\big), \tag{2.4}$$

where

$$\rho\det\mathtt{F} = 1, \tag{2.5}$$

and

$$S_{ij}(\mathtt{F}) = \frac{\partial W}{\partial F_{ij}}. \tag{2.6}$$

Here S, $\rho S\mathtt{F}^\top$, $W(\mathtt{F})$ denote *Piola–Kirchhoff stress*, *Cauchy stress* and the elastic energy of the material respectively. Recall that the condition (2.6) implies that the material is called hyperelastic [20]. In the case of Hookean (linear) elasticity [13,14,16],

$$W(\mathtt{F}) = \frac{1}{2}|\mathtt{F}|^2 = \frac{1}{2}\,\mathrm{tr}\big(\mathtt{F}\mathtt{F}^\top\big), \tag{2.7}$$

where the notation tr stands for the trace operator of a matrix, and hence,

$$S(\mathtt{F}) = \mathtt{F}. \tag{2.8}$$

Combining Eqs. (2.1)–(2.8) together, we obtain system (1.1). In particular, if the density ρ is a constant, the equations of the incompressible viscoelastic flows have the following form (see [2,12–14,16,21] and references therein):

$$\begin{cases} \operatorname{div}\mathbf{u} = 0, \\ \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \operatorname{div}\big(\mathtt{F}\mathtt{F}^\top\big), \\ \partial_t\mathtt{F} + \mathbf{u}\cdot\nabla\mathtt{F} = \nabla\mathbf{u}\,\mathtt{F}. \end{cases} \tag{2.9}$$

3. Local existence

In this section, we will prove the existence part in Theorem 1.1. The proof will proceed through four steps by combining the Lax–Milgram theorem and a fixed-point argument. To this end, we consider first the linearized problem.

Set

$$\mathcal{W}' = \big\{\psi \in \big(L^2\big(0,T;H^2\big(\mathbb{R}^3\big)\big)\big)^3\colon\ \partial_t\psi \in L^2(Q_T)\big\}$$

with the natural norm $\|\psi\|_{\mathcal{W}'}$, and for $q \in (3,6]$ we define

$$\begin{aligned} \mathcal{W} = \mathcal{W}' \cap \big(L^2\big(0,T;W^{2,q}\big(\mathbb{R}^3\big)\big) \cap L^\infty\big(0,T;H^2\big(\mathbb{R}^3\big)\big)\big)^3 \\ \cap \big\{\psi \in \big(L^2\big(0,T;H^2\big(\mathbb{R}^3\big)\big)\big)^3\colon\ \psi_t \in L^\infty\big(0,T;L^2\big(\mathbb{R}^3\big)\big),\ \nabla\psi_t \in L^2(Q_T),\ \psi(0) = \mathbf{u}_0\big\}. \end{aligned}$$

Consider the following linearized problem:

$$\partial_t\rho + \operatorname{div}(\rho v) = 0, \tag{3.1a}$$

$$\rho\partial_t\mathbf{u} - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla\operatorname{div}\mathbf{u} = -\rho v\cdot\nabla v - \nabla P + \operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big), \tag{3.1b}$$

$$\partial_t\mathtt{F} + v\cdot\nabla\mathtt{F} = \nabla v\mathtt{F}. \tag{3.1c}$$

with the given $v \in \mathcal{W}$ and the initial condition (1.2).

3.1. Solvability of the density with a fixed velocity

Let $A_j(x,t)$, $j = 1,\dots,n$, be symmetric $m\times m$ matrices, $B(x,t)$ an $m\times m$ matrix, $f(x,t)$ and $V_0(x)$ two m-dimensional vector functions defined in $\mathbb{R}^n\times(0,T)$ and $\mathbb{R}^n$, respectively.

For the Cauchy problem of the linear system on $V \in \mathbb{R}^m$:

$$\begin{cases} \partial_t V + \sum_{j=1}^{n} A_j(x,t)\partial_{x_j} V + B(x,t)V = f(x,t), \\ V(x,0) = V_0(x), \end{cases} \tag{3.2}$$

we have

Lemma 3.1. *Assume that*

$$A_j \in \left[C\left(0,T; H^s(\mathbb{R}^n)\right) \cap C^1\left(0,T; H^{s-1}(\mathbb{R}^n)\right)\right]^{m\times m}, \quad j = 1,\dots,n,$$
$$B \in C\left((0,T), H^{s-1}(\mathbb{R}^n)\right)^{m\times m}, \qquad f \in C\left((0,T), H^s(\mathbb{R}^n)\right)^m, \qquad V_0 \in H^s(\mathbb{R}^n)^m,$$

with $s > \frac{n}{2}+1$ an integer. Then there exists a unique solution to (3.2)*, i.e, a function*

$$V \in \left[C\left([0,T), H^s(\mathbb{R}^n)\right) \cap C^1\left((0,T), H^{s-1}(\mathbb{R}^n)\right)\right]^m$$

satisfying (3.2) *pointwise* (*i.e. in the classical sense*).

Proof. This lemma is a direct consequence of Theorem 2.16 in [24] with $A_0(x,t) = I$. $\square$

To solve the density with respect to the velocity, we have

Lemma 3.2. *Under the same conditions as Theorem* 1.1*, there is a unique strictly positive function*

$$\rho := \mathcal{S}(v) \in W^{1,2}\left(0,T; L^q(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)\right) \cap L^\infty\left(0,T; W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)\right)$$

which satisfy the continuity Eq. (3.1a)*. Moreover, the density satisfies the following estimate*:

$$\|\nabla\rho\|^q_{L^\infty(0,T;L^q(\mathbb{R}^3))\cap L^2(\mathbb{R}^3)} \leqslant \left(\|\nabla\rho_0\|^q_{L^q(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)} + \sqrt{T}\|v\|_{\mathcal{W}}\right)\exp\left(C\sqrt{T}\|v\|_{\mathcal{W}}\right).$$

Here, and in what follows, the notation C stands for a generic positive constant, and in some cases, we will specify its dependence on parameters by the notation $C(\cdot)$; and $W^{1,2}(0,T;X) = \{f\colon\ f, f_t \in L^2(0,T,X)\}$. In many cases below, we drop the constant C for the simplicity of notations.

Proof. The proof of the first part of this lemma is similar to that of Lemma 3.3 below, and can also be found in Theorem 9.3 in [24]. The positivity of the density follows directly from the observation: by writing (3.1a) along characteristics $\frac{d}{dt}X(t) = v$,

$$\frac{d}{dt}\rho\left(t, X(t)\right) = -\rho\left(t, X(t)\right)\operatorname{div} v\left(t, X(t)\right), \qquad X(0) = x,$$

and with the help of Gronwall's inequality,

$$\alpha\exp\left(-\sqrt{T}\|v\|_{\mathcal{W}}\right) \leqslant \left(\inf_x \rho_0\right)\exp\left(-\int_0^t \left\|\operatorname{div} v(t)\right\|_{L^\infty(\mathbb{R}^3)}ds\right) \leqslant \rho(t,x)$$
$$\leqslant \left(\sup_x \rho_0\right)\exp\left(\int_0^t \left\|\operatorname{div} v(t)\right\|_{L^\infty(\mathbb{R}^3)}ds\right) \leqslant \beta\exp\left(\sqrt{T}\|v\|_{\mathcal{W}}\right).$$

Now, we can assume that the continuity equation holds pointwise in the following form

$$\partial_t \rho + \rho \operatorname{div} v + v \cdot \nabla \rho = 0.$$

Taking the gradient in both sides of the above identity, multiplying by $|\nabla \rho|^{q-2} \nabla \rho$ and then integrating over $\mathbb{R}^3$, we get, by Young's inequality,

$$\begin{aligned}
\frac{1}{q} \frac{d}{dt} \|\nabla \rho\|^q_{L^q(\mathbb{R}^3)} &\leqslant \int_{\mathbb{R}^3} |\nabla \rho|^q |\operatorname{div} v|\, dx + \int_{\mathbb{R}^3} \rho |\nabla \rho|^{q-1} |\nabla \operatorname{div} v|\, dx + \int_{\mathbb{R}^3} |\nabla v||\nabla \rho|^q\, dx \\
&\quad - \frac{1}{q} \int_{\mathbb{R}^3} v \nabla |\nabla \rho|^q\, dx \\
&\leqslant \|\nabla \rho\|^q_{L^q} \bigl(\|\nabla v\|_{L^\infty} + \|\rho\|_{L^\infty} \|\nabla \operatorname{div} v\|_{L^q} \bigr) + \frac{1}{q} \int_{\mathbb{R}^3} \operatorname{div} v |\nabla \rho|^q\, dx \\
&\quad + \|\rho\|_{L^\infty} \|\nabla \operatorname{div} v\|_{L^q} \\
&\leqslant C \|\nabla \rho\|^q_{L^q} \|v\|_{W^{2,q}} + \|\rho\|_{L^\infty} \|\nabla \operatorname{div} v\|_{L^q},
\end{aligned} \tag{3.3}$$

since $W^{1,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ as $q > 3$. Using Gronwall's inequality, we conclude that

$$\begin{aligned}
\bigl\|\nabla \rho(t)\bigr\|^q_{L^q(\mathbb{R}^3)} &\leqslant \left(\|\nabla \rho_0\|^q_{L^q} + \int_0^t \|\rho\|_{L^\infty} \|\nabla \operatorname{div} v\|_{L^q}\, ds \right) \exp\left(\int_0^t \|v\|_{W^{2,q}}\, ds \right) \\
&\leqslant \bigl(\|\nabla \rho_0\|^q_{L^q} + \sqrt{t} \|v\|_{\mathcal{W}} \bigr) \exp\bigl(\sqrt{t} \|v\|_{\mathcal{W}} \bigr).
\end{aligned} \tag{3.4}$$

The proof is complete. $\square$

3.2. Solvability of the deformation gradient with a fixed velocity

Due to the hyperbolic structure of (3.1c), we can apply Lemma 3.1 again to solve the deformation gradient $\mathtt{F}$ in terms of the given velocity. For this purpose, we have

Lemma 3.3. *Under the same conditions as Theorem* 1.1, *there is a unique function*

$$\mathtt{F} := \mathcal{T}(v) \in W^{1,2}\bigl(0, T; L^q\bigl(\mathbb{R}^3\bigr) \cap L^2\bigl(\mathbb{R}^3\bigr)\bigr) \cap L^\infty\bigl(0, T; W^{1,q}\bigl(\mathbb{R}^3\bigr) \cap W^{1,2}\bigl(\mathbb{R}^3\bigr)\bigr)$$

which satisfies Eq. (3.1c)*. Moreover, the deformation gradient satisfies*

$$\|\mathtt{F}\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3)) \cap H^1(\mathbb{R}^3)} \leqslant \bigl(\bigl\|\mathtt{F}(0)\bigr\|_{W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)} + \sqrt{T} \|v\|_{\mathcal{W}} \bigr) \exp\bigl(\sqrt{T} \|v\|_{\mathcal{W}} \bigr).$$

Proof. First, we assume that $v \in C^1(0, T; C_0^\infty(\mathbb{R}^3))$, $\mathtt{F}_0 \in C_0^\infty(\mathbb{R}^3)$. Then, we can rewrite (3.1c) in the component form of columns as

$$\partial_t \mathtt{F}_k + v \cdot \nabla \mathtt{F}_k = \nabla v \mathtt{F}_k, \quad \text{for all } 1 \leqslant k \leqslant 3.$$

Applying Lemma 3.1 with $A_j(x,t) = v_j(x,t)I$ for $1 \leqslant j \leqslant 3$, $B(x,t) = -\nabla v$, and $f(x,t) = 0$, we get a solution

$$\mathtt{F} \in \bigcap_{s=3}^{\infty} \bigl\{ C^1\bigl(0,T,H^{s-1}\bigl(\mathbb{R}^3\bigr)\bigr) \cap C\bigl(0,T;H^s\bigl(\mathbb{R}^3\bigr)\bigr)\bigr\}.$$

This implies, by the Sobolev embedding theorems,

$$\mathtt{F} \in \bigcap_{k=1}^{\infty} C^1\bigl(0,T;C^k\bigl(\mathbb{R}^3\bigr)\bigr) = C^1\bigl(0,T;C^\infty\bigl(\mathbb{R}^3\bigr)\bigr).$$

Next, for $v \in \mathcal{W}$, by an argument of dense sets, there is a sequence of functions v_n in the space $C^1(0,T;C_0^\infty(\mathbb{R}^3))$, $v_n \to v$ in $\mathcal{W}$, and $\mathtt{F}_0^n \in C_0^\infty(\mathbb{R}^3)$, $\mathtt{F}_0^n \to \mathtt{F}_0$ in $W^{1,q}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. Hence, $v_n \to v$ in $C(B(0,a) \times (0,T))$ for all $a > 0$ where $B(0,a)$ denotes the ball with radius a and centered at the origin. According to the previous result, there are $\{\mathtt{F}_n\}_{n=1}^\infty$ satisfying

$$\partial_t \mathtt{F}_n + v_n \cdot \nabla \mathtt{F}_n = \nabla v_n \mathtt{F}_n, \tag{3.5}$$

with $\mathtt{F}_n(0) = \mathtt{F}_0^n$, $\mathtt{F}_n \in C^1(0,T;C^\infty(\mathbb{R}^3))$. Multiply (3.5) by $|\mathtt{F}_n|^{p-2}\mathtt{F}_n$ for any $p \geqslant 2$, and integrating over $\mathbb{R}^3$, by integration by parts, we obtain, using Young's inequality,

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3} |\mathtt{F}_n|^p\,dx &= -\frac{1}{p}\int_{\mathbb{R}^3} v_n \cdot \nabla |\mathtt{F}_n|^p\,dx + \int_{\mathbb{R}^3} \nabla v_n |\mathtt{F}_n|^{p-2}\mathtt{F}_n^2\,dx \\ &\leqslant \frac{1+p}{p}\|\mathtt{F}_n\|_{L^p}^p \|\nabla v_n\|_{L^\infty}. \end{aligned}$$

Then, by Gronwall's inequality, one has

$$\begin{aligned} \int_{\mathbb{R}^3} |\mathtt{F}_n|^p\,dx(t) &\leqslant \int_{\mathbb{R}^3} \bigl|\mathtt{F}_n(0)\bigr|^p\,dx \exp\left(\int_0^t (p+1)\|\nabla v_n\|_{L^\infty}\,ds\right) \\ &\leqslant \int_{\mathbb{R}^3} \bigl|\mathtt{F}_n(0)\bigr|^p\,dx \exp\left(\int_0^t (p+1)\|v_n\|_{W^{2,p}}\,ds\right). \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathtt{F}_n\|_{L^\infty(0,T;L^p(\mathbb{R}^3))} &\leqslant \exp\left(\frac{p+1}{p}\|v_n\|_{\mathcal{W}}\sqrt{t}\right)\bigl\|\mathtt{F}_n(0)\bigr\|_{L^p(\mathbb{R}^3)} \\ &\leqslant \exp\bigl(2\|v_n\|_{\mathcal{W}}\sqrt{t}\bigr)\bigl\|\mathtt{F}_n(0)\bigr\|_{L^p(\mathbb{R}^3)} < \infty. \end{aligned}$$

Letting $p \to \infty$, one obtains

$$\|\mathtt{F}_n\|_{L^\infty(Q_T)} \leqslant \exp\bigl(\|v_n\|_{\mathcal{W}}\sqrt{t}\bigr)\bigl\|\mathtt{F}_n(0)\bigr\|_{L^\infty(\mathbb{R}^3)} \leqslant \exp\bigl(\|v_n\|_{\mathcal{W}}\sqrt{t}\bigr)\bigl\|\mathtt{F}_n(0)\bigr\|_{W^{1,q}(\mathbb{R}^3)} < \infty.$$

Hence, up to a subsequence, we can assume that v_n were chosen so that

$$\mathtt{F}_n \to \mathtt{F} \quad \text{weak-}^* \text{ in } L^\infty\big(0,T;L^q\big(\mathbb{R}^3\big)\big).$$

Taking the gradient in both sides of (3.5), multiplying by $|\nabla \mathtt{F}_n|^{q-2}\nabla \mathtt{F}_n$ and then integrating over $\mathbb{R}^3$, we get, with the help of Hölder's inequality and Young's inequality,

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|\nabla \mathtt{F}_n\|^q_{L^q(\mathbb{R}^3)} \\
&\quad\leqslant 2\int_{\mathbb{R}^3}|\nabla \mathtt{F}_n|^q|\nabla v_n|\,dx+\int_{\mathbb{R}^3}|\mathtt{F}_n||\nabla \mathtt{F}_n|^{q-1}|\nabla\nabla v_n|\,dx-\frac{1}{q}\int_{\mathbb{R}^3}v_n\nabla|\nabla \mathtt{F}_n|^q\,dx \\
&\quad\leqslant C\int_{\mathbb{R}^3}|\nabla \mathtt{F}_n|^q|\nabla v_n|\,dx+\int_{\mathbb{R}^3}|\mathtt{F}_n||\nabla \mathtt{F}_n|^{q-1}|\nabla\nabla v_n|\,dx \\
&\quad\leqslant C\|\nabla \mathtt{F}_n\|^q_{L^q}\|v_n\|_{W^{2,q}}+\|\mathtt{F}_n\|_{L^\infty}\|v_n\|_{W^{2,q}}\|\nabla \mathtt{F}_n\|^{q-1}_{L^q} \\
&\quad\leqslant C\|\nabla \mathtt{F}_n\|^q_{L^q}\|v_n\|_{W^{2,q}}+C\|v_n\|_{W^{2,q}}\|\nabla \mathtt{F}_n\|^{q-1}_{L^q},
\end{aligned} \tag{3.6}$$

since $q>3$. Using Gronwall's inequality, we conclude that

$$\begin{aligned}
\big\|\nabla \mathtt{F}_n(t)\big\|_{L^q(\mathbb{R}^3)} &\leqslant \left(\big\|\nabla \mathtt{F}_n(0)\big\|_{L^q}+\int_0^t\|v_n\|_{W^{2,q}}\,ds\right)\exp\left(\int_0^t\|v_n\|_{W^{2,q}}\,ds\right) \\
&\leqslant \big(\big\|\nabla \mathtt{F}_n(0)\big\|_{L^q}+\sqrt{t}\|v_n\|_{\mathcal{W}}\big)\exp\big(\sqrt{t}\|v_n\|_{\mathcal{W}}\big),
\end{aligned}$$

and hence,

$$\begin{aligned}
\|\nabla \mathtt{F}\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} &\leqslant \liminf_{n\to\infty}\|\nabla \mathtt{F}_n\|_{L^\infty(0,T;L^q)} \\
&\leqslant \big(\big\|\nabla \mathtt{F}(0)\big\|_{L^q}+\sqrt{T}\|v\|_{\mathcal{W}}\big)\exp\big(\sqrt{T}\|v\|_{\mathcal{W}}\big).
\end{aligned} \tag{3.7}$$

Thus,

$$\|\mathtt{F}\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^3))} \leqslant \big(\big\|\mathtt{F}(0)\big\|_{W^{1,q}}+\sqrt{T}\|v\|_{\mathcal{W}}\big)\exp\big(\sqrt{T}\|v\|_{\mathcal{W}}\big)<\infty.$$

Passing to the limit as $n\to\infty$ in (3.5), we show that (3.1c) holds at least in the sense of distributions. Therefore, $\partial_t \mathtt{F}\in L^2(0,T;L^2(\mathbb{R}^3))$, then $\mathtt{F}\in W^{1,2}(0,T;L^q(\mathbb{R}^3)\cap L^2(\mathbb{R}^3))$. The proof is complete. $\square$

3.3. *Local solvability of (3.1b)*

For simplicity of the presentation, we consider the case $\mu=1$ and $\lambda=0$ without loss of generality. In order to solve (3.1b), we consider the bilinear form $E(\mathbf{u},\psi)$ and linear functional $L(\psi)$ defined by

$$\begin{aligned}
E(\mathbf{u},\psi) &= \int_0^T\big(\rho\partial_t\mathbf{u}-\Delta\mathbf{u}-\nabla\operatorname{div}\mathbf{u},\,\partial_t\psi-k(\Delta\psi+\nabla\operatorname{div}\psi)\big)\,dt \\
&\quad-\big(\mathbf{u}(0),\,\Delta\psi(0)+\nabla\operatorname{div}\psi(0)\big),
\end{aligned}$$

$$\begin{aligned} L(\psi) = & -\int_0^T \big(\rho v \cdot \nabla v + \nabla P - \operatorname{div}\big(\rho \mathtt{F}\mathtt{F}^\top\big), \partial_t \psi - k(\Delta\psi + \nabla \operatorname{div}\psi)\big)\, dt \\ & - \big(\mathbf{u}_0, \Delta\psi(0) + \nabla \operatorname{div}\psi(0)\big), \end{aligned}$$

with

$$k = \big(2\|\rho\|_{L^\infty(Q_T)}\big)^{-1}$$

for $\psi \in \mathcal{W}'$, where $(\cdot,\cdot)$ denotes the inner product in L^2.

We first notice that $L(\psi)$ is a linear continuous functional of ψ with respect to the norm $\|\psi\|_{\mathcal{W}'}$. Moreover, we have

$$\begin{aligned} & E(\psi,\psi) \\ & = \int_0^T \big(\|\sqrt{\rho}\partial_t\psi\|_{L^2}^2 + k\|\Delta\psi + \nabla \operatorname{div}\psi\|_{L^2}^2 - k(\rho\partial_t\psi, \Delta\psi + \nabla \operatorname{div}\psi)\big)\, dt \\ & \quad + \frac{1}{2}\big(\big\|\nabla\psi(T)\big\|_{L^2}^2 + \big\|\nabla\psi(0)\big\|_{L^2}^2 + \big\|\operatorname{div}\psi(T)\big\|_{L^2}^2 + \big\|\operatorname{div}\psi(0)\big\|_{L^2}^2\big) \\ & \geqslant \int_0^T \bigg(\|\sqrt{\rho}\partial_t\psi\|_{L^2}^2 + k\|\Delta\psi + \nabla \operatorname{div}\psi\|_{L^2}^2 - \frac{3}{4}\|\sqrt{\rho}\partial_t\psi\|_{L^2}^2 - \frac{2k}{3}\|\Delta\psi + \nabla \operatorname{div}\psi\|_{L^2}^2\bigg)\, dt \\ & \quad + \frac{1}{2}\big(\big\|\nabla\psi(T)\big\|_{L^2}^2 + \big\|\nabla\psi(0)\big\|_{L^2}^2 + \big\|\operatorname{div}\psi(T)\big\|_{L^2}^2 + \big\|\operatorname{div}\psi(0)\big\|_{L^2}^2\big) \\ & \geqslant c_0\|\psi\|_{\mathcal{W}'}^2, \end{aligned}$$

for some $c_0 > 0$, since

$$\|\Delta\psi + \nabla \operatorname{div}\psi\|_{L^2} \geqslant c_0\|\psi\|_{H^2}$$

from the theory of elliptic operators. Hence, by the Lax–Milgram theorem (see [5]), there exists a $\mathbf{u} \in \mathcal{W}'$ such that

$$E(\mathbf{u},\psi) = L(\psi) \tag{3.8}$$

for every $\psi \in \mathcal{W}'$.

Now, let $\overline{\psi}$ be the solution of the problem

$$\begin{gathered} \partial_t\overline{\psi} - k(\Delta\overline{\psi} + \nabla \operatorname{div}\overline{\psi}) = 0, \\ \overline{\psi}(0) = h(x), \end{gathered}$$

with $h(x)$ smooth enough. Replacing in (3.8) ψ by $\overline{\psi}$, one obtain

$$\big(\mathbf{u}(0) - \mathbf{u}_0, \Delta h + \nabla \operatorname{div} h\big) = 0,$$

which implies $\mathbf{u}(0)=\mathbf{u}_0$. Next, let $\widetilde{\psi}$ be a solution of the following problem

$$\partial_t\widetilde{\psi}-k(\Delta\widetilde{\psi}+\nabla\operatorname{div}\widetilde{\psi})=g(x,t),$$
$$\widetilde{\psi}(0)=0,$$

with g smooth enough. Replacing ψ by $\widetilde{\psi}$ in (3.8), one obtain

$$\int_0^T\big(\rho\partial_t\mathbf{u}-\Delta\mathbf{u}-\nabla\operatorname{div}\mathbf{u}+\rho v\cdot\nabla v-\operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big)+\nabla P,g\big)\,dt=0.$$

This implies that $(\mathbf{u},\rho,\mathtt{F})$ satisfies (3.1) a.e. in $(0,T)\times\mathbb{R}^3$.

Next, we prove the higher regularity for $\mathbf{u}$; that is, $\mathbf{u}\in\mathcal{W}$. First, we multiply (3.1b) by $\partial_t\mathbf{u}$, and use integration by parts and Young's inequality to obtain

$$\begin{aligned}
&\frac{2}{3}\int_0^t\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}^2\,ds+\frac{1}{2}\big\|\nabla\mathbf{u}(t)\big\|_{L^2}^2+\frac{1}{2}\big\|\operatorname{div}\mathbf{u}(t)\big\|_{L^2}^2\\
&\leqslant\frac{1}{2}\big\|\nabla\mathbf{u}(0)\big\|_{L^2}^2+\frac{1}{2}\big\|\operatorname{div}\mathbf{u}(0)\big\|_{L^2}^2-\int_0^t(\nabla P,\partial_t\mathbf{u})\,ds\\
&\quad+C\sup_{s\in(0,t)}\|\rho\|_{L^\infty}\int_0^t\|v\|_{L^6}^2\|\nabla v\|_{L^3}^2\,ds+\int_0^t\big(\operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big),\partial_t\mathbf{u}\big)\,ds+\frac{1}{4}\int_0^t\|\partial_t\nabla\mathbf{u}\|_{L^2}^2\,ds\\
&\leqslant\frac{1}{2}\big\|\nabla\mathbf{u}(0)\big\|_{L^2}^2+\frac{1}{2}\big\|\operatorname{div}\mathbf{u}(0)\big\|_{L^2}^2+\int_0^t\|\nabla\rho\|_{L^2}^2\,ds+\frac{1}{4}\int_0^t\|\partial_t\nabla\mathbf{u}\|_{L^2}^2\,ds\\
&\quad+C\sup_{s\in(0,t)}\|\rho\|_{L^\infty}\int_0^t\|v\|_{L^6}^2\|\nabla v\|_{L^3}^2\,ds+\int_0^t\|\nabla\mathtt{F}\|_{L^2}^2\,ds+\frac{1}{3}\int_0^t\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}^2\,ds.
\end{aligned}\tag{3.9}$$

In particular, if we multiply (3.1b) by $\partial_t\mathbf{u}$, integrate over $\mathbb{R}^3$, and let $t=0$, we obtain

$$\begin{aligned}
\big\|\sqrt{\rho}(0)\partial_t\mathbf{u}(0)\big\|_{L^2}^2&=\int_{\mathbb{R}^3}\big(\Delta\mathbf{u}(0)+\nabla\operatorname{div}\mathbf{u}(0)-\rho(0)v(0)\cdot\nabla v(0)-\nabla P\big(\rho(0)\big)\\
&\quad+\operatorname{div}\big(\rho(0)\mathtt{F}(0)\mathtt{F}^\top(0)\big)\big)\partial_t\mathbf{u}(0)\,dx\\
&\leqslant\big\|\sqrt{\rho(0)}\,\big\|_{L^\infty}^{-1}\big(\|\Delta\mathbf{u}_0\|_{L^2}+\big\|\rho(0)\big\|_{L^\infty}\big\|v(0)\big\|_{L^\infty}\big\|\nabla v(0)\big\|_{L^2}\\
&\quad+C\big\|\nabla\rho(0)\big\|_{L^2}+\big\|\nabla\mathtt{F}(0)\big\|_{L^2}\big)\big\|\sqrt{\rho}(0)\partial_t\mathbf{u}(0)\big\|_{L^2},
\end{aligned}$$

which implies that

$$\big\|\sqrt{\rho}(0)\partial_t\mathbf{u}(0)\big\|_{L^2}\leqslant C\big(\|\Delta\mathbf{u}_0\|_{L^2}+\|v_0\|_{L^\infty}\|\nabla v_0\|_{L^2}+C\|\nabla\rho_0\|_{L^2}+\|\nabla\mathtt{F}_0\|_{L^2}\big).\tag{3.10}$$

Now we differentiate (3.1b) with respect to t so that we get

$$\begin{aligned}
&\partial_t\rho\partial_t\mathbf{u}+\rho\partial_{tt}^2\mathbf{u}-\Delta\partial_t\mathbf{u}-\nabla\operatorname{div}(\partial_t\mathbf{u})\\
&\quad=-\partial_t\rho v\cdot\nabla v-\rho\partial_t v\cdot\nabla v-\rho v\cdot\nabla\partial_t v-\nabla\partial_t P+\partial_t\operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big).
\end{aligned}\tag{3.11}$$

Multiplying (3.11) by $\partial_t\mathbf{u}$, integrating over $\mathbb{R}^3$, and using the continuity equation, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}^3}\partial_t\rho|\partial_t\mathbf{u}|^2\,dx+\|\nabla\partial_t\mathbf{u}\|_{L^2}^2+\|\operatorname{div}\partial_t\mathbf{u}\|_{L^2}^2\\
&\quad\leqslant\|\rho\|_{L^\infty}\|v\|_{L^\infty}\|\nabla v\|_{L^2}\|\nabla v\|_{L^3}\|\partial_t\mathbf{u}\|_{L^6}+\|\rho\|_{L^\infty}\|v\|_{L^6}^2\big\|\nabla(\nabla v)\big\|_{L^2}\|\partial_t\mathbf{u}\|_{L^6}\\
&\qquad+\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t v\|_{L^2}\|\nabla v\|_{L^3}\|\partial_t\mathbf{u}\|_{L^6}+\|\rho\|_{L^\infty}\|v\|_{L^\infty}\|v\|_{L^6}\|\nabla v\|_{L^3}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\
&\qquad+\|\sqrt{\rho}\|_{L^\infty}\|\nabla\partial_t v\|_{L^2}\|v\|_{L^\infty}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}+\|\partial_t P\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\\
&\qquad+\|\nabla\partial_t\mathbf{u}\|_{L^2}\big(\|\nabla\rho\|_{L^2}\|v\|_{L^\infty}\|\mathtt{F}\|_{L^\infty}^2+\|\rho\|_{L^\infty}\|\nabla v\|_{L^2}\|\mathtt{F}\|_{L^\infty}^2\\
&\qquad+\|\rho\|_{L^\infty}\|\mathtt{F}\|_{L^\infty}\|v\|_{L^\infty}\|\nabla\mathtt{F}\|_{L^2}\big).
\end{aligned}\tag{3.12}$$

Integrating (3.12) with respect to t, using the continuity equation and the Gagliardo–Nirenberg inequality

$$\|\nabla\mathbf{u}\|_{L^3}^2\leqslant C\|\nabla\mathbf{u}\|_{L^2}\|\Delta\mathbf{u}\|_{L^2},$$

we find, since $\rho\in L^\infty((0,T)\times\mathbb{R}^3)$,

$$\begin{aligned}
&\frac{1}{2}\big\|\sqrt{\rho}\partial_t\mathbf{u}(t)\big\|_{L^2}^2+\int_0^t\big(\|\nabla\partial_t\mathbf{u}\|_{L^2}^2+\|\operatorname{div}\partial_t\mathbf{u}\|_{L^2}^2\big)\,ds\\
&\quad\leqslant\frac{1}{2}\big\|\sqrt{\rho}\partial_t\mathbf{u}(0)\big\|_{L^2}^2\\
&\qquad+\int_0^t\|\rho\|_{L^\infty}\big(\|v\|_{L^\infty}\|\nabla v\|_{L^2}^{\frac{3}{2}}\|\Delta v\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t\mathbf{u}\|_{L^2}+\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\big)\,ds\\
&\qquad+\int_0^t\|\sqrt{\rho}\|_{L^\infty}\big(\|\sqrt{\rho}\partial_t v\|_{L^2}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\Delta\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t\mathbf{u}\|_{L^2}+\|v\|_{L^\infty}\|\nabla\partial_t\mathbf{u}\|_{L^2}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}\big)\,ds\\
&\qquad+\int_0^t\|\sqrt{\rho}\|_{L^\infty}\|\nabla\partial_t v\|_{L^2}\|v\|_{L^\infty}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}\,ds+\int_0^t\|\partial_t P\|_{L^2}\|\nabla\partial_t\mathbf{u}\|_{L^2}\,ds\\
&\qquad+C_1\int_0^t\|\nabla\partial_t\mathbf{u}\|_{L^2}\big(\|v\|_{L^\infty}+\|\nabla v\|_{L^2}\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\big\|\sqrt{\rho}\partial_t\mathbf{u}(0)\big\|_{L^2}^2 + C_\delta\int_0^t \|v\|_{L^\infty}^2\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}^2\,ds \\
&\quad + C_\delta\int_0^t \big(\|v\|_{L^\infty}^2\|\nabla v\|_{L^2}^3\|\Delta v\|_{L^2} + C_\delta\|\nabla v\|_{L^2}^4\|\Delta v\|_{L^2}^2\big)\,ds \\
&\quad + C_\delta\int_0^t \|\sqrt{\rho}\partial_t v\|_{L^2}^2\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}\,ds + C_\delta\int_0^t \big\|\operatorname{div}(\rho v)\big\|_{L^2}^2\,ds \\
&\quad + \delta\int_0^t \big(\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + \|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}^2\big)\,ds + C_\delta\int_0^t \big(\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^2}^2\big)\,ds + \int_0^t \|\nabla\partial_t v\|_{L^2}^2\,ds, \qquad (3.13)
\end{aligned}
$$

where δ is a small constant, C_1 depends on v, ρ_0, and $\mathtt{F}_0$, since

$$
\nabla\rho \in L^\infty\big(0,T;L^q\big(\mathbb{R}^3\big)\cap L^2\big(\mathbb{R}^3\big)\big), \qquad \nabla\mathtt{F} \in L^\infty\big(0,T;L^q\big(\mathbb{R}^3\big)\cap L^2\big(\mathbb{R}^3\big)\big)
$$

as stated in Lemma 3.2 and Lemma 3.3. Since we are only interested in the local existence, so we can restrict $t \leqslant \overline{T} \leqslant 1$.

Adding (3.9) and (3.13) for some suitable $\delta < \frac{1}{2}$, one obtains, first by Gronwall's inequality,

$$
\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^\infty(0,t;L^2(\mathbb{R}^3))} \leqslant C,
$$

and, second,

$$
\begin{aligned}
&\frac{1}{2}\big\|\sqrt{\rho}\partial_t\mathbf{u}(t)\big\|_{L^2}^2 + \frac{1}{4}\int_0^t \big(\|\nabla\partial_t\mathbf{u}\|_{L^2}^2 + \|\operatorname{div}\partial_t\mathbf{u}\|_{L^2}^2\big)\,ds + \frac{1}{3}\int_0^t \|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^2}^2\,ds \\
&\quad + \frac{1}{2}\big\|\nabla\mathbf{u}(t)\big\|_{L^2}^2 + \frac{1}{2}\big\|\operatorname{div}\mathbf{u}(t)\big\|_{L^2}^2 \leqslant C,
\end{aligned}
$$

which implies

$$
\begin{cases}
\sqrt{\rho}\partial_t\mathbf{u} \in L^\infty\big(0,T;L^2\big(\mathbb{R}^3\big)\big); \quad \partial_t\mathbf{u} \in L^2\big(0,T;H^1\big(\mathbb{R}^3\big)\big); \\
\nabla\mathbf{u} \in L^\infty\big(0,T;L^2\big(\mathbb{R}^3\big)\big).
\end{cases} \qquad (3.14)
$$

On the other hand, we can rewrite (3.1b) as

$$
-\Delta\mathbf{u} - \nabla\operatorname{div}\mathbf{u} = -\rho\partial_t\mathbf{u} - \rho v\cdot\nabla v - \nabla P + \operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big),
$$

which is a strongly elliptic equation. Hence, by the classical theory in the elliptic system, we deduce that

$$
\mathbf{u} \in L^\infty\big(0,T;H^2\big(\mathbb{R}^3\big)\big),
$$

since from Lemmas 3.2–3.3 and (3.14) one has

$$
-\rho\partial_t\mathbf{u} - \rho v\cdot\nabla v - \nabla P + \operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big) \in L^\infty\big(0,T;L^2\big(\mathbb{R}^3\big)\big).
$$

Moreover, since ρ is bounded from below and $\sqrt{\rho}\partial_t\mathbf{u} \in L^2(0,T;L^2(\mathbb{R}^3))$, we know that $\partial_t\mathbf{u} \in L^2(0,T;L^2(\mathbb{R}^3))$. Hence, by the Gagliardo–Nirenberg inequality, as $q \in (3,6]$,

$$\|\partial_t\mathbf{u}\|_{L^2(0,T;L^q(\mathbb{R}^3))} \leqslant C\|\partial_t\mathbf{u}\|^{\theta}_{L^2((0,T)\times\mathbb{R}^3)}\|\nabla\partial_t\mathbf{u}\|^{1-\theta}_{L^2((0,T)\times\mathbb{R}^3)},$$

for some $\theta \in [0,1)$. This implies that, by (3.14), $\partial_t\mathbf{u} \in L^2(0,T;L^q(\mathbb{R}^3))$. Thus, by the classical elliptic theory, we obtain $\mathbf{u} \in L^2(0,T;W^{2,q}(\mathbb{R}^3))$ since now

$$-\rho\partial_t\mathbf{u} - \rho v\cdot\nabla v - \nabla P + \operatorname{div}\big(\rho \mathtt{F}\mathtt{F}^\top\big) \in L^2\big(0,T;L^q\big(\mathbb{R}^3\big)\big).$$

Hence, we can conclude that $\mathbf{u} \in \mathcal{W}$.

3.4. Existence for (1.1)

The above argument leads us to define the map

$$\mathbf{u} = \mathcal{H}(v)$$

from $\mathcal{W}$ to itself through the maps $g : v \mapsto \mathcal{S}(v)$, $f : v \mapsto \mathcal{T}(v)$ and $d : (\mathcal{S}(v), v, \mathcal{T}(v)) \mapsto \mathbf{u}$. Hence the solution of (1.1) is obtained from a fixed point of the map $\mathcal{H}$. To find a fixed point of $\mathcal{H}$, we will use the Schauder–Tychonoff fixed point theorem (Theorem 5.28, [28]). Define

$$\begin{aligned} M = \big\{\psi\colon\ \max\big(&\|\psi\|_{L^2(0,T;W^{2,q}(\mathbb{R}^3)\cap H^2(\mathbb{R}^3))}, \|\sqrt{\rho}\partial_t\psi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))},\\ &\|\psi\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}, \|\partial_t\psi\|_{L^2(0,T;H^1(\mathbb{R}^3))}\big) \leqslant r\big\}, \end{aligned}$$

where

$$r = C_0\big(\|\mathbf{u}_0\|_{H^2} + \|\mathbf{u}_0\|_{L^\infty}\|\nabla\mathbf{u}_0\|_{L^2} + \|\rho_0\|_{W^{1,q}\cap H^1} + \|\mathtt{F}_0\|_{W^{1,q}\cap H^1}\big)$$

with some sufficiently large $C_0 > 0$. Clearly, M is a compact and convex set in $L^2((0,T)\times\mathbb{R}^3)$. Hence, we need to show that $\mathcal{H}(M) \subseteq M$ (i.e., $\mathcal{H}$ maps M into M) and $\mathcal{H}$ is continuous in M with respect to the norm in $L^2((0,T)\times\mathbb{R}^3)$.

We first prove that $\mathcal{H}(M) \subset M$ for some $T = \bar{T}$. Indeed, assuming $v \in M$, from Lemma 3.2 and Lemma 3.3, we know that

$$\begin{cases} \alpha\exp(-r\sqrt{t}) \leqslant \rho(x,t) \leqslant \beta\exp(r\sqrt{t});\\ \|\rho\|_{L^\infty(0,t;W^{1,q}(\mathbb{R}^3))} \leqslant \big(\|\rho_0\|_{W^{1,q}(\mathbb{R}^3)} + \sqrt{t}r\big)\exp(\sqrt{t}r);\\ \|\mathtt{F}\|_{L^\infty(0,t;W^{1,q}(\mathbb{R}^3))} \leqslant \big(\big\|\mathtt{F}(0)\big\|_{W^{1,q}(\mathbb{R}^3)} + \sqrt{t}r\big)\exp(\sqrt{t}r). \end{cases} \tag{3.15}$$

Hence, from (3.13) and (3.15), it follows that

$$\begin{aligned} &\frac{1}{2}\big\|\sqrt{\rho}\partial_t\mathbf{u}(t)\big\|^2_{L^2} + \frac{1}{2}\int_0^t \|\nabla\partial_t\mathbf{u}\|^2_{L^2}\,ds\\ &\quad\leqslant \frac{1}{2}\big\|\sqrt{\rho}\partial_t\mathbf{u}(0)\big\|^2_{L^2} + C_\delta t\big(r^2+\delta\big)\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^\infty(0,t;L^2(\mathbb{R}^3))} + C_\delta t\big(r^4+r^6\big) + Ctr^2. \end{aligned}$$

Using (3.10), and taking δ and $\overline{T}$ sufficiently small, we derive from the above inequality that

$$\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^\infty(0,\overline{T};L^2(\mathbb{R}^3))}+\|\nabla\partial_t\mathbf{u}\|^2_{L^2(0,\overline{T};L^2(\mathbb{R}^3))}\leqslant\frac{1}{3}r^2.$$

Since ρ is bounded from below, we obtain

$$\|\sqrt{\rho}\partial_t\mathbf{u}\|^2_{L^\infty(0,\overline{T};L^2(\mathbb{R}^3))}+\|\partial_t\mathbf{u}\|^2_{L^2(0,\overline{T};H^1(\mathbb{R}^3))}\leqslant\frac{2}{3}r^2. \tag{3.16}$$

To estimate the norm $\|\mathbf{u}\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$, noticing that

$$-\Delta\mathbf{u}-\nabla\operatorname{div}\mathbf{u}=-\rho\partial_t\mathbf{u}-\rho v\cdot\nabla v-\nabla P(\rho)+\operatorname{div}\big(\rho\mathtt{F}\mathtt{F}^\top\big),$$

one has from the theory for elliptic equations that

$$\begin{aligned}\|\mathbf{u}\|_{L^\infty(0,\overline{T};H^2(\mathbb{R}^3))}\leqslant{}&\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t\mathbf{u}\|_{L^\infty(0,\overline{T};L^2)}+\|\rho\|_{L^\infty}\|v\|_{L^\infty}\|\nabla v\|_{L^\infty(0,\overline{T};L^2)}\\&+C\|\nabla\rho\|_{L^\infty(0,\overline{T};L^2)}+\|\nabla\mathtt{F}\|_{L^\infty(0,\overline{T};L^2)};\end{aligned}$$

and hence

$$\|\Delta\mathbf{u}+\nabla\operatorname{div}\mathbf{u}\|^2_{L^\infty(0,\overline{T};H^2)}\leqslant r^2.$$

Next, we need to obtain estimates on $\|\mathbf{u}\|_{L^2(0,T;W^{2,q}(\mathbb{R}^3))}$. Indeed, we have, by the classical theory of elliptic equations,

$$\begin{aligned}\int_0^{\overline{T}}\|\mathbf{u}\|^2_{W^{2,q}}\,dt&\leqslant C\int_0^{\overline{T}}\big(\|v\|^2_{L^\infty}\|\nabla v\|^2_{L^q}+\|\nabla\rho\|^2_{L^q}+\|\partial_t\mathbf{u}\|^2_{L^q}+\|\nabla\mathtt{F}\|^2_{L^q}\big)\,dt\\&\leqslant Cr^4\overline{T}+r^2\overline{T}\leqslant r^2,\end{aligned}$$

for some sufficiently small $\overline{T}$. Hence, we show that $\mathcal{H}(M)\subseteq M$.

Finally, we need to prove the continuity of $\mathcal{H}$ in M. First we observe that if $\{v_n\}_{n=1}^\infty\subseteq M$, then there exists a subsequence (still denoted by $\{v_n\}_{n=1}^\infty$) such that as $n\to\infty$, $v_n\to v$ strongly in M. Let ρ_n and ρ be the solutions of

$$\partial_t\rho_n+\operatorname{div}(\rho_n v_n)=0,$$

and

$$\partial_t\rho+\operatorname{div}(\rho v)=0,$$

with $\rho_n(0)=\rho(0)=\rho_0$, respectively. Denoting $\overline{\rho_n}=\rho_n-\rho$, then $\overline{\rho_n}$ satisfies

$$\partial_t\overline{\rho_n}+v_n\cdot\nabla\overline{\rho_n}+(v_n-v)\cdot\nabla\rho+\overline{\rho_n}\operatorname{div}v_n+\rho\operatorname{div}(v_n-v)=0,$$

with $\overline{\rho_n}(0)=0$. Repeating the argument in Lemma 3.2, we have

$$\|\overline{\rho_n}\|^2_{L^2}\leqslant\exp(Cr\overline{T})\int_0^{\overline{T}}\big(\big\|(v-v_n)\cdot\nabla\rho\big\|^2_{L^2}+\big\|\rho\operatorname{div}(v_n-v)\big\|^2_{L^2}\big)\,dt,$$

which implies that $\rho_n \to \rho$ strongly in $L^\infty(0, \overline{T}; L^2(\mathbb{R}^3))$. Similarly, we can show that $\mathtt{F}_n \to \mathtt{F}$ strongly in $L^\infty(0, \overline{T}; L^2(\mathbb{R}^3))$. Now let $\mathbf{u}_n$ and $\mathbf{u}$ be the solutions of (3.1b) corresponding to v_n and v with $\mathbf{u}_n(0) = \mathbf{u}(0) = \mathbf{u}_0$ respectively. Then one has, denoting $U_n = \mathbf{u}_n - \mathbf{u}$ and $V_n = v_n - v$,

$$\rho_n \partial_t U_n - \Delta U_n - \nabla \operatorname{div} U_n = -\overline{\rho_n} \partial_t \mathbf{u} - \rho_n V_n \cdot \nabla v_n - \overline{\rho_n} v \cdot \nabla v_n - \rho v \cdot \nabla V_n \\ + \operatorname{div}\big(\rho_n \mathtt{F}_n \mathtt{F}_n^\top - \rho \mathtt{F}\mathtt{F}^\top\big) - \nabla P(\rho_n) + \nabla P(\rho).$$

Multiplying the above equation by $\partial_t U_n$, integrating over $(0, \overline{T}) \times \mathbb{R}^3$, and thanking to the convergence of ρ_n and $\mathtt{F}_n$, we can prove as a routine matter that $\nabla U_n \to 0$ strongly in $L^2((0, \overline{T}) \times \mathbb{R}^3)$ and $\sqrt{\rho_n} \partial_t U_n \to 0$ strongly in $L^2((0, \overline{T}) \times \mathbb{R}^3)$. Due to the lower bound of ρ_n, we deduce that $\partial_t U_n \to 0$ strongly in $L^2((0, \overline{T}) \times \mathbb{R}^3)$, and hence $U_n \to 0$ in $L^2((0, \overline{T}) \times \mathbb{R}^3)$ by using the identity $U_n(t) = \int_0^t \partial_t U_n \, ds$ since $U_n(0) = 0$. Thus, the map $\mathcal{H}$ is continuous in M. The existence of a local solution is completely proved.

4. Uniqueness

In this section, we will prove the uniqueness of the solution obtained in the previous section. Notice that, the argument in Section 3 yields that

$$\partial_t \mathbf{u} \in L^2\big(0, T; L^2 \cap L^q\big(\mathbb{R}^3\big)\big), \qquad \nabla \rho \in L^2\big(0, T; L^2 \cap L^q\big(\mathbb{R}^3\big)\big), \qquad \nabla \mathtt{F} \in L^2\big(0, T; L^2 \cap L^q\big(\mathbb{R}^3\big)\big),$$

for $q > 3$. Hence, using the interpolation, we see that

$$\partial_t \mathbf{u} \in L^2\big(0, T; L^3\big(\mathbb{R}^3\big)\big), \qquad \nabla \rho \in L^2\big(0, T; L^3\big(\mathbb{R}^3\big)\big), \qquad \nabla \mathtt{F} \in L^2\big(0, T; L^3\big(\mathbb{R}^3\big)\big).$$

Now, assume that $\mathbf{u}_1$, $\mathbf{u}_2$ satisfy (1.1) for some $T > 0$, and let

$$r := \mathcal{S}(\mathbf{u}_1) - \mathcal{S}(\mathbf{u}_2), \qquad v := \mathbf{u}_1 - \mathbf{u}_2, \qquad G := \mathcal{T}(\mathbf{u}_1) - \mathcal{T}(\mathbf{u}_2).$$

Then, we have

$$\begin{cases} \partial_t r + \mathbf{u}_1 \cdot \nabla r + v \cdot \nabla \mathcal{S}(\mathbf{u}_2) + r \operatorname{div} \mathbf{u}_1 + \mathcal{S}(\mathbf{u}_2) \operatorname{div} v = 0, \\ r(0) = 0. \end{cases} \tag{4.1}$$

Multiplying (4.1) by r, and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{2} \frac{d}{dt} \|r\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^3} |r|^2 \operatorname{div} \mathbf{u}_1 \, dx + \int_{\mathbb{R}^3} v \nabla \mathcal{S}(\mathbf{u}_2) r \, dx \\ + \int_{\mathbb{R}^3} |r|^2 \operatorname{div} \mathbf{u}_1 \, dx + \int_{\mathbb{R}^3} r \mathcal{S}(\mathbf{u}_2) \operatorname{div} v \, dx = 0,$$

which yields

$$\frac{d}{dt} \|r\|_{L^2(\mathbb{R}^3)}^2 \leqslant \| \operatorname{div} \mathbf{u}_1 \|_{L^\infty} \|r\|_{L^2}^2 + \varepsilon \|\nabla v\|_{L^2}^2 + C(\varepsilon) \big\| \nabla \mathcal{S}(\mathbf{u}_2) r \big\|_{L^{\frac{6}{5}}}^2 \\ + \varepsilon \|\nabla v\|_{L^2}^2 + C(\varepsilon) \big\| \mathcal{S}(\mathbf{u}_2) \big\|_{L^\infty}^2 \|r\|_2^2$$

$$\begin{aligned}
&\leqslant \|\operatorname{div}\mathbf{u}_1\|_{L^\infty}\|r\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\nabla\mathcal{S}(\mathbf{u}_2)\big\|_{L^3}^2\|r\|_{L^2}^2\\
&\quad+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\mathcal{S}(\mathbf{u}_2)\big\|_{L^\infty}^2\|r\|_2^2\\
&\leqslant \eta_1(\varepsilon)\|r\|_{L^2}^2+2\varepsilon\|\nabla v\|_{L^2}^2,
\end{aligned}\tag{4.2}$$

where $\eta_1(\varepsilon)=\|\operatorname{div}\mathbf{u}_1\|_{L^\infty}+C(\varepsilon)(\|\nabla\mathcal{S}(\mathbf{u}_2)\|_{L^3}^2+\|\mathcal{S}(\mathbf{u}_2)\|_{L^\infty}^2)$.

Similarly, from (3.1c), we obtain

$$\begin{cases}\partial_t G+\mathbf{u}_1\cdot\nabla G+v\cdot\nabla\mathcal{T}(\mathbf{u}_2)=\nabla\mathbf{u}_1 G+\nabla v\mathcal{T}(\mathbf{u}_2),\\ G(0)=0.\end{cases}\tag{4.3}$$

Multiplying (4.3) by G, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2-\frac{1}{2}\int_{\mathbb{R}^3}|G|^2\operatorname{div}\mathbf{u}_1\,dx+\int_{\mathbb{R}^3}v\cdot\nabla\mathcal{T}(\mathbf{u}_2):G\,dx\\
&\quad=\int_{\mathbb{R}^3}G^\top\nabla\mathbf{u}_1 G\,dx+\int_{\mathbb{R}^3}\nabla v\mathcal{T}(\mathbf{u}_2):G\,dx,
\end{aligned}$$

which yields

$$\begin{aligned}
\frac{d}{dt}\|G\|_{L^2(\mathbb{R}^3)}^2&\leqslant \|\operatorname{div}\mathbf{u}_1\|_{L^\infty}\|G\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\nabla\mathcal{T}(\mathbf{u}_2)G\big\|_{L^{\frac{6}{5}}}^2\\
&\quad+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\mathcal{T}(\mathbf{u}_2)\big\|_{L^\infty}^2\|G\|_2^2\\
&\leqslant \|\operatorname{div}\mathbf{u}_1\|_{L^\infty}\|G\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\nabla\mathcal{T}(\mathbf{u}_2)\big\|_{L^3}^2\|G\|_{L^2}^2\\
&\quad+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\mathcal{T}(\mathbf{u}_2)\big\|_{L^\infty}^2\|G\|_2^2\\
&\leqslant \eta_2(\varepsilon)\|G\|_{L^2}^2+2\varepsilon\|\nabla v\|_{L^2}^2,
\end{aligned}\tag{4.4}$$

where $\eta_2(\varepsilon)=\|\operatorname{div}\mathbf{u}_1\|_{L^\infty}+C(\varepsilon)(\|\nabla\mathcal{T}(\mathbf{u}_2)\|_{L^3}^2+\|\mathcal{T}(\mathbf{u}_2)\|_{L^\infty}^2)$.

For each $\mathbf{u}_j$, $j=1,2$, we deduce from (3.1b) that

$$\begin{cases}\mathcal{S}(\mathbf{u}_j)\partial_t\mathbf{u}_j-\mu\Delta\mathbf{u}_j-(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}_j\\ \quad=-\mathcal{S}(\mathbf{u}_j)(\mathbf{u}_j\cdot\nabla)\mathbf{u}_j-\nabla P\big(\mathcal{S}(\mathbf{u}_j)\big)+\operatorname{div}\big(\mathcal{S}(\mathbf{u}_j)\mathcal{T}(\mathbf{u}_j)\mathcal{T}(\mathbf{u}_j)^\top\big),\\ \mathbf{u}_j(0)=\mathbf{u}_0.\end{cases}$$

Subtracting these equations, we obtain,

$$\begin{aligned}
&\mathcal{S}(\mathbf{u}_1)\partial_t\mathbf{u}_1-\mathcal{S}(v_2)\partial_t\mathbf{u}_2-\mu\Delta v-(\mu+\lambda)\nabla\operatorname{div}v\\
&\quad=-\mathcal{S}(\mathbf{u}_1)(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1+\mathcal{S}(\mathbf{u}_2)(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2-\nabla P\big(\mathcal{S}(\mathbf{u}_1)\big)+\nabla P\big(\mathcal{S}(\mathbf{u}_2)\big)\\
&\qquad+\operatorname{div}\big(\mathcal{S}(\mathbf{u}_1)\mathcal{T}(\mathbf{u}_1)\mathcal{T}(\mathbf{u}_1)^\top\big)-\operatorname{div}\big(\mathcal{S}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)^\top\big).
\end{aligned}\tag{4.5}$$

Since

$$\begin{aligned}
&-\mathcal{S}(\mathbf{u}_1)(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1+\mathcal{S}(\mathbf{u}_2)(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\\
&\quad=-\mathcal{S}(\mathbf{u}_1)(v\cdot\nabla)\mathbf{u}_1-\big(\mathcal{S}(\mathbf{u}_1)-\mathcal{S}(\mathbf{u}_2)\big)(\mathbf{u}_2\cdot\nabla)\mathbf{u}_1-\mathcal{S}(\mathbf{u}_2)(\mathbf{u}_2\cdot\nabla)v,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{S}(\mathbf{u}_1)\mathcal{T}(\mathbf{u}_1)\mathcal{T}(\mathbf{u}_1)^\top-\mathcal{S}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)^\top\\
&\quad=\mathcal{S}(\mathbf{u}_1)G\mathcal{T}(\mathbf{u}_1)^\top+r\mathcal{T}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_1)^\top+\mathcal{S}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)G^\top,
\end{aligned}$$

we can rewrite (4.5) as

$$\begin{aligned}
&\mathcal{S}(\mathbf{u}_1)\partial_t v-\mu\Delta v-(\mu+\lambda)\nabla\operatorname{div}v\\
&\quad=-r\partial_t\mathbf{u}_2-\mathcal{S}(\mathbf{u}_1)(v\cdot\nabla)\mathbf{u}_1-r(\mathbf{u}_2\cdot\nabla)\mathbf{u}_1-\mathcal{S}(\mathbf{u}_2)(\mathbf{u}_2\cdot\nabla)v-\nabla P\big(\mathcal{S}(\mathbf{u}_1)\big)+\nabla P\big(\mathcal{S}(\mathbf{u}_2)\big)\\
&\qquad+\operatorname{div}\big(\mathcal{S}(\mathbf{u}_1)G\mathcal{T}(\mathbf{u}_1)^\top+r\mathcal{T}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_1)^\top+\mathcal{S}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)G^\top\big).
\end{aligned}\tag{4.6}$$

Multiplying (4.6) by v, using the continuity equation (1.1a) and integrating over $\mathbb{R}^3$, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\mathcal{S}(\mathbf{u}_1)|v|^2\,dx+\int_{\mathbb{R}^3}\big(\mu|\nabla v|^2+(\lambda+\mu)|\operatorname{div}v|^2\big)\,dx\\
&\quad=\int_{\mathbb{R}^3}\bigg\{\frac{1}{2}\mathcal{S}(\mathbf{u}_1)(\mathbf{u}_1\cdot\nabla)v\cdot v-r\partial_t\mathbf{u}_2 v-\mathcal{S}(\mathbf{u}_1)(v\cdot\nabla)\mathbf{u}_1 v-r(\mathbf{u}_2\cdot\nabla)\mathbf{u}_1 v\\
&\qquad-\mathcal{S}(\mathbf{u}_2)(\mathbf{u}_2\cdot\nabla)v v-\nabla P\big(\mathcal{S}(\mathbf{u}_1)\big)v+\nabla P\big(\mathcal{S}(\mathbf{u}_2)\big)v\\
&\qquad-\big(\mathcal{S}(\mathbf{u}_1)G\mathcal{T}(\mathbf{u}_1)^\top+r\mathcal{T}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_1)^\top+\mathcal{S}(\mathbf{u}_2)\mathcal{T}(\mathbf{u}_2)G^\top\big)\nabla v\bigg\}\,dx\\
&\quad\leqslant\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\mathcal{S}(\mathbf{u}_1)\big\|_{L^\infty}^2\|\mathbf{u}_1\|_{L^\infty}^2\|v\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\|\partial_t\mathbf{u}_2\|_{L^3}^2\|r\|_{L^2}^2\\
&\qquad+\big\|\mathcal{S}(\mathbf{u}_1)\big\|_{L^\infty}\|\nabla\mathbf{u}_1\|_{L^\infty}\|v\|_{L^2}^2+2\|\mathbf{u}_2\|_{L^\infty}\|\nabla\mathbf{u}_1\|_{L^\infty}\big(\|r\|_{L^2}^2+\|v\|_{L^2}^2\big)\\
&\qquad+\varepsilon\|\nabla v\|_{L^2}^2+C(\varepsilon)\big\|\mathcal{S}(\mathbf{u}_2)\big\|_{L^\infty}^2\|\mathbf{u}_2\|_{L^\infty}^2\|v\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2\\
&\qquad+C(\varepsilon)\big(\sup\big\{P'(\eta)\colon C(T)^{-1}\leqslant\eta\leqslant C(T)\big\}\big)^2\|r\|_{L^2}^2+\varepsilon\|\nabla v\|_{L^2}^2\\
&\qquad+C(\varepsilon)\Big(\big\|\mathcal{S}(\mathbf{u}_1)\big\|_{L^\infty}^2\big\|\mathcal{T}(\mathbf{u}_1)\big\|_{L^\infty}^2\|G\|_{L^2}^2\\
&\qquad+\big\|\mathcal{S}(\mathbf{u}_2)\big\|_{L^\infty}^2\big\|\mathcal{T}(\mathbf{u}_2)\big\|_{L^\infty}^2\|G\|_{L^2}^2+\|r\|_{L^2}^2\big\|\mathcal{T}(\mathbf{u}_1)\big\|_{L^\infty}\big\|\mathcal{T}(\mathbf{u}_2)\big\|_{L^\infty}^2\Big)\\
&\quad\leqslant 5\varepsilon\|\nabla v\|_{L^2}^2+\eta_3(\varepsilon)\big(\|r\|_{L^2}^2+\|v\|_{L^2}^2+\|G\|_{L^2}^2\big)
\end{aligned}\tag{4.7}$$

with

$$\begin{aligned}
\eta_3(\varepsilon)&=C(\varepsilon)\big\|\mathcal{S}(\mathbf{u}_1)\big\|_{L^\infty}^2\|\mathbf{u}_1\|_{L^\infty}^2+C(\varepsilon)\|\partial_t\mathbf{u}_2\|_{L^3}^2+\big\|\mathcal{S}(\mathbf{u}_1)\big\|_{L^\infty}\|\nabla\mathbf{u}_1\|_{L^\infty}\\
&\quad+2\|\mathbf{u}_2\|_{L^\infty}\|\nabla\mathbf{u}_1\|_{L^\infty}+C(\varepsilon)\big\|\mathcal{S}(\mathbf{u}_2)\big\|_{L^\infty}^2\|\mathbf{u}_2\|_{L^\infty}^2
\end{aligned}$$

$$
\begin{aligned}
&+ C(\varepsilon)\big(\sup\big\{P'(\eta):\ C(T)^{-1} \leqslant \eta \leqslant C(T)\big\}\big)^2 \\
&+ C(\varepsilon)\big(\big\|\mathcal{S}(\mathbf{u}_1)\big\|_{L^\infty}^2 \big\|\mathcal{T}(\mathbf{u}_1)\big\|_{L^\infty}^2 + \big\|\mathcal{S}(\mathbf{u}_2)\big\|_{L^\infty}^2 \big\|\mathcal{T}(\mathbf{u}_2)\big\|_{L^\infty}^2\big) \\
&+ \big\|\mathcal{T}(\mathbf{u}_1)\big\|_{L^\infty}^2 \big\|\mathcal{T}(\mathbf{u}_2)\big\|_{L^\infty}^2.
\end{aligned}
$$

Summing up (4.2), (4.4), and (4.7), by taking $\varepsilon = \frac{\mu}{20}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \big(\mathcal{S}(v_1)|v|^2 + |r|^2 + |G|^2\big)\,dx + \mu \int_{\mathbb{R}^3} |\nabla v|^2\,dx \\
&\leqslant 2\big(\eta_3(\varepsilon) + \eta_2(\varepsilon) + \eta_1(\varepsilon)\big)\big(\|v\|_{L^2}^2 + \|r\|_{L^2}^2 + \|G\|_{L^2}^2\big) \\
&\leqslant 2\eta(\varepsilon, t) \int_{\mathbb{R}^3} \big(\mathcal{S}(\mathbf{u}_1)|v|^2 + |r|^2 + |G|^2\big)\,dx,
\end{aligned}
\tag{4.8}
$$

with

$$
\eta(\varepsilon, t) = \frac{\eta_3(\varepsilon) + \eta_2(\varepsilon) + \eta_1(\varepsilon)}{\min\{\min_{x\in\mathbb{R}^3} \mathcal{S}(\mathbf{u}_1)(x,t), 1\}}.
$$

It is a routine matter to establish the integrability with respect to t of the function $\eta(\varepsilon, t)$ on the interval $(0, T)$. This is a consequence of the regularity of $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{W}$ and the estimates in Lemmas 3.2 and 3.3 for $\mathcal{S}(\mathbf{u}_i)$, $\mathcal{T}(\mathbf{u}_i)$ with $i = 1, 2$. Therefore, (4.8), combining with Gronwall's inequality, implies

$$
\int_{\mathbb{R}^3} \big(\mathcal{S}(\mathbf{u}_1)|v|^2 + |r|^2 + |G|^2\big)\,dx = 0, \quad \text{for all } t \in (0, T),
$$

and consequently

$$
v \equiv 0, \qquad r \equiv 0, \qquad G \equiv 0.
$$

Thus, the proof of uniqueness is complete.

Acknowledgments

Xianpeng Hu's research was supported in part by the National Science Foundation grant DMS-0604362 and by the Mellon Predoctoral Fellowship at the University of Pittsburgh. Dehua Wang's research was supported in part by the National Science Foundation under Grants DMS-0604362 and DMS-0906160, and by the Office of Naval Research under Grant N00014-07-1-0668.

References

[1] J. Chemin, N. Masmoudi, About lifespan of regular solutions of equations related to viscoelastic fluids, SIAM J. Math. Anal. 33 (2001) 84–112.
[2] Y. Chen, P. Zhang, The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions, Comm. Partial Differential Equations 31 (2006) 1793–1810.
[3] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, second ed., Grundlehren Math. Wiss., vol. 325, Springer-Verlag, Berlin, 2005.
[4] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford Lecture Ser. Math. Appl., vol. 26, Oxford University Press, Oxford, 2004.
[5] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics Math., Springer-Verlag, Berlin, 2001.

[6] G.B. Goktekin, A.W. Bargteil, J.F. O'Brien, A method for animating viscoelastic fluids, ACM Trans. Graphics 23 (2004) 463–468.
[7] D. Wang, Strong solutions to the three-dimensional compressible viscoelastic fluids, preprint.
[8] F. John, Blow-up for quasilinear wave equations in three space dimensions, Comm. Pure Appl. Math. 34 (1981) 29–51.
[9] D. Joseph, Fluid Dynamics of Viscoelastic Liquids, Appl. Math. Sci., vol. 84, Springer-Verlag, New York, 1990.
[10] P. Kessenich, Global existence with small initial data for three-dimensional incompressible isotropic viscoelastic materials, preprint.
[11] K. Kunisch, M. Marduel, Optimal control of non-isothermal viscoelastic fluid flow, J. Non-Newtonian Fluid Mechanics 88 (2000) 261–301.
[12] F. Lin, C. Liu, P. Zhang, On hydrodynamics of viscoelastic fluids, Comm. Pure Appl. Math. 58 (2005) 1437–1471.
[13] Z. Lei, C. Liu, Y. Zhou, Global existence for a 2D incompressible viscoelastic model with small strain, Commun. Math. Sci. 5 (2007) 595–616.
[14] Z. Lei, C. Liu, Y. Zhou, Global solutions for incompressible viscoelastic fluids, Arch. Ration. Mech. Anal. 188 (2008) 371–398.
[15] Z. Lei, C. Liu, Y. Zhou, Global solutions for compressible viscoelastic fluids with small initial data, in preparation.
[16] F. Lin, P. Zhang, On the initial-boundary value problem of the incompressible viscoelastic fluid system, Comm. Pure Appl. Math. 61 (2008) 539–558.
[17] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1. Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 3, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.
[18] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2. Compressible Models, Oxford Lecture Ser. Math. Appl., vol. 10, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.
[19] P.L. Lions, N. Masmoudi, Global solutions for some Oldroyd models of non-Newtonian flows, Chin. Ann. Math. Ser. B 21 (2000) 131–146.
[20] C. Liu, N.J. Walkington, An Eulerian description of fluids containing visco-elastic particles, Arch. Ration. Mech. Anal. 159 (2001) 229–252.
[21] Z. Lei, Y. Zhou, Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit, SIAM J. Math. Anal. 37 (2005) 797–814.
[22] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980) 67–104.
[23] A. Matsumura, T. Nishida, Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Comm. Math. Phys. 89 (1983) 445–464.
[24] A. Novotný, I. Straškraba, Introduction to the Mathematical Theory of Compressible Flow, Oxford Lecture Ser. Math. Appl., vol. 27, Oxford University Press, Oxford, 2004.
[25] J.G. Oldroyd, On the formation of rheological equations of state, Proc. R. Soc. Lond. Ser. A 200 (1950) 523–541.
[26] J.G. Oldroyd, Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids, Proc. R. Soc. Lond. Ser. A 245 (1958) 278–297.
[27] M. Renardy, W.J. Hrusa, J.A. Nohel, Mathematical Problems in Viscoelasticity, Longman Scientific and Technical; copublished in the US with John Wiley, New York, 1987.
[28] W. Rubin, Functional Analysis, McGraw–Hill, 1991.
[29] R. Salvi, I. Straškraba, Global existence for viscous compressible fluids and their behavior as $t \to \infty$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 40 (1993) 17–51.
[30] T.C. Sideris, B. Thomases, Global existence for three-dimensional incompressible isotropic elastodynamics via the incompressible limit, Comm. Pure Appl. Math. 58 (2005) 750–788.
[31] T.C. Sideris, Nonlinear hyperbolic systems and elastodynamics, in: Phase Space Analysis of Partial Differential Equations, vol. II, in: Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa, 2004, pp. 451–485.
[32] J.D. Yu, S. Sakai, J.A. Sethian, Two-phase viscoelastic jetting, J. Comput. Phys. 220 (2007) 568–585.